UDC 519.85:517.957

On Algorithm of Integrability Classification of the Nonlinear Dynamical Systems via Computer Algebra Methods

Bohdan Fil[1], Yaroslav Pelekh[2], Myroslava Vovk[3], Halyna Beregova[4], Tatiana Magerovska[5], Pavlo Pukach[6]

[1] Dr.Phil., Doc., Department of Computational Mathematics and Programming Lviv Polytechnic National University, 12 Bandera street, Lviv, Ukraine, 79013, e-mail: bohdan.m.fil@lpnu.ua
[2] Dr.Phil., Doc., Department of Computational Mathematics and Programming Lviv Polytechnic National University , 12 Bandera street, Lviv, Ukraine, 79013, e-mail: Pelekh_Ya_M@ukr.net
[3] Dr.Phil., Doc., Department of Higher Mathematics Lviv Polytechnic National University, 12 Bandera street, Lviv, Ukraine, 79013, e-mail: myroslava.i.vovk@lpnu.ua
[4] Dr.Phil., Doc., Department of Computational Mathematics and Programming Lviv Polytechnic National University, 12 Bandera street, Lviv, Ukraine, 79013, e-mail: gberegova@yahoo.com
[5] Dr.Phil., Doc.,Department of Computational Mathematics and Programming Lviv Polytechnic National University, 12 Bandera street, Lviv, Ukraine, 79013, e-mail: magerovskat@gmail.com
[6] Department of Artifical Intelligence Systems Lviv Polytechnic National University, 12 Bandera street, Lviv, Ukraine, 79013, e-mail: pavlopukach@gmail.com

There is developed an algorithm to classify integrable nonlinear dynamical systems via Wolfram Mathematica. The hierarchy of conservation laws for the nonlinear dynamical system can be calculated by this algorithm. There are demonstrated some modifications of nonlinear Korteweg-de Vries equations integrated by inverse scatering method.

Keywords: Computer algebra system; Wolfram Mathematica; Nonlinear wave equation; Conservation laws; Integrability nonlinear dynamical systems; Conserved energy; KdV equation.

Introduction. It is well known that solving the nonlinear partial equations is quite complicated procedure and moreover, the result can't be ensured. That's why it is important to find instruments that can propose quick answer for the question "Are these solving methods reasonable for application?" Method basing on existence of conservation laws of the initial dynamical system [1] can be used.

Namely it is application of gradient-holonomic algorithm that makes possible to calculate hierarchy (in general case infinite) of conservation laws being attached to the algebraic structures. Due to these algebraic structures nonlinear dynamic system can be represented as Lax pair and, in consequence, method of inverse dissipation problem to find solutions of the initial dynamical system [2] can be used. At the same time the solutions can be calculated. Procedure of constructing of conservation laws was already realized by computer algebra methods. Sanders together with Roelof [3] (1994) and Wang [4] (1995) developed software package for Maple and FORM, using intension of differential operators into Heisenberg Algebra.

Wolf developed the package Con-Law 1-4 [5] (1999) in REDUCE language that was based on the solution of overdetermined partial differential equations system. This system needs some additional parameters to be calculated. Göktas Ü., and Hereman W. [6] (1996) developed package condes.m for Mathematica that calculates polynomial conservation laws for the nonlinear polynomial partial differential equations. Ruo-Xia Yao Zhi-Bin Li [7] (2006) using Maple developed package CONSLAW, that can find conservation laws via the undetermined coefficients method.

Existence of nondegenerated hierarchy of conservation laws makes available to calculate the invariant structures of nonlinear dynamical systems [8, 9, 10, 11, 12]. Calculation of symmetry algebras for the nonlinear dynamical system can be also realized by analogical methods. Fil B. realized the conservation laws search for the nonlinear dynamical system in language REDUCE [13] (1991). Integro-differential conservation laws for the nonlinear system with four equations were found (modified vector variant of Schrödinger equations, its integrability was studied by M. Prytula). Hence in REDUCE (at that time) there wasn't realized the effective algorithm to find integrals of undetermined functions, this part of work was done by hand.

This question of nonlinear dynamical system integrability is still of current interest. As a mater of course it must be realized by computer-integrated methods taking into account that nonlinear differential equations often serve an instrument of study for proffesionals not only in mathematics. That is the aim to realize some elements of gradient holonomic method in computer algebra system in this paper. SCA Mathematica was chosen to describe this procedure.

1. Formulation of the problem

At first, let's present shortly the mathematical part of algorithm [10]. Let's consider the nonlinear dynamical system

$$du/dt = K[u] \ , \tag{1}$$

where $t \in \mathrm{R}$, $u := u(x,\mathrm{t}) \in \mathrm{M} \subset \mathrm{C}^{\infty}\left(\mathrm{R},\mathrm{R}^{\mathrm{m}}\right)$, $\mathrm{K}[\mathrm{u}]$ is vector field in form of the differential equation on the functions u). The equation solution

$$d\phi/dt + K'^{*} \cdot \phi = 0, \tag{2}$$

$$\phi \cong b(x;\lambda)\exp\left[\partial^{-1}\sigma(x;\lambda) + \omega(x,t;\lambda)\right], \tag{3}$$

$$\left(b(x;\lambda) \cong \sum\nolimits_{j\in \mathrm{Z}_+} b_j[u]\lambda^{-j+\tilde{b}}, \ \ \sigma(x;\lambda) \cong \sum\nolimits_{j\in \mathrm{Z}_+} \sigma_j[u]\lambda^{-j+\tilde{\sigma}} \ , \ |\lambda| \to \infty \right).$$

Generates conservation laws $\gamma(\lambda) := \int_0^{2\pi} \sigma(x\ \lambda)dx$, where $\tilde{b},\ \tilde{\sigma} \in \mathrm{Z}_+$ are some fixed values (they would be zero in our case), $\omega(x,t;\lambda)$ is dispersion function, that can calculated explicitly, under condition that (2), doesn't depend on u. Existence of nondegenerated infinite sequence of conservation laws $\gamma_j := \int_0^{2\pi} \sigma_j[u]dx \ \ j \in \mathrm{Z}_+$, is necessary condition of dynamical system integrability via the inverse scatering method [10]. Taking into account that equations (1), (2) with the relationships (3) generate recursive equations on functions $\sigma_j[u]$, one can calculate in sequence functions, generating conservations laws. If sequence of conservation laws $\sigma_j[u]$ is trivial (beginning from some number the next are total derivatives), or, while calculating one can obtain $\sigma_k[u]$ (arbitrary *k*), not being conservation law of system (1), then system (1) is not intergrable via inverse scatering method.

For example, for Korteweg-de Vries (KdV) system $u_t = -u_{xxx} + 6uu_x := K[u]$ using this algorithm one can get $d\phi / dt + K'^* \cdot \phi = 0, \left(K'^* = \partial^3 - 6u\partial\right),$ $\phi = \exp\left(\partial^{-1}\sigma(u;\lambda) - \lambda^3 t + \lambda x\right)$. There are calculated recurrence relationships from the equation $3\lambda^2\sigma + \lambda\left(3\sigma_x + 3\sigma^2 - 6u\right) + \partial_t \int_0^x \sigma dx + \sigma_{xx} + 2\sigma\sigma_x + \sigma^3 - 6u\sigma = 0$.

Solving them with $\sigma_1, \sigma_3, \sigma_5, \ldots$, one can obtain nontrivial conservation laws

$$\gamma_1 = \int_0^{2\pi} -udx,\ \gamma_3 = \int_0^{2\pi} -2u^2 dx,\ \gamma_5 = \int_0^{2\pi} \left(-2u_x^2 + u^3\right) dx.$$

2. Solving task using Wolfram Mathematica

Let's describe set of commands Mathematica to calculate conservation laws in case of one equation (there is demonstrated example for calculation conservation laws for KdV system). To estimate the dynamical system integrability it is sufficient to choose conservation laws quantity twice or triple than the system order in practice. So we'll interrupt infinite expansions preseting desired number.

Let's specify the initial equation (right-hand side (1)), desired conservation laws quantity to be calculated and order of differential equation on x:

```
ut[x_,t_]:=-D[u[x,t],{x,3}]+ 6*u[x,t]*D[u[x,t],x]↵ kilk:=10↵ por:=3
```

Calculating K'^* :

```
KFrSpr[w_]:=Sum[(-1)^k*D[D[ut[x,t], u(k,0)[x,t]]*w, {x,k}], {k,0,por}]
```

Subsidiary function:

```
u(n_,1)[x,t]:=D[ut[x,t],{x,n}]
```

calculation φ:

```
φ[x_,t_]:=Exp[Sum[Integrate[φ[i][x,t],x]*λ^i,{i,0,kilk+por}]+x/λ+ (-1)^
(por+1)* Coefficient[ut[x,t],D[u[x,t],{x,por}]]*t/λ^por]
```

Recurrence relationships on densities of conservation laws σ_i are the next:

```
RekRivn=CoefficientList[Collect[Simplify[Expand[(D[φ[x,t],t]+ KFr-
Spr[φ[x,t]])/ φ[x,t]]]*λ^por,λ],λ];
```

Whence one can calculate the densities of conservation laws:

```
Do[σ[i][x_,t_]=Replace[σ[i][x,t],Solve[RekRivn[[2+i]]==0, σ[i][x,t]]][[1]],
{i,0,kilk}].
```

Then let's check generation of conservation laws by these densities . If there is number of σ_i and 0, that means that this density generates conservation law. And conversely, nonzero value means that conversation law is not generated and consequently, system is not integrable via inverse scatering method:

```
Do[Print[{k,Simplify[Sum[(-1)^i* D[D[D[σ[k][x,t],t], u(i,0)[x,t], {x,i}]],
{i,0,kilk}]]}],{k,0,kilk}]
```

Output of conservation laws (the first number is an order of conservation law):

```
Do[Print[{i,Simplify[σ[i][x,t]]}],{i,0,kilk}]
```

The last step: is σ_i the total derivative? (there is presented pair: density number and 0, if it is total derivative):

Do[Print[{k,Simplify[Sum[(-1)^i*D[D[σ[k][x,t], u(i,0)[x,t],{x,i}]], {i,0,kilk}]]}], {k,0,kilk}]

Application of given operator sequence to Korteweg-de Vries system proposes 10 the first conservation laws:

{**0**, 0}↵ {**1**,2u[x,t]}↵ {**2**, -2u(1,0)[x,t]}↵ {**3**, -2u[x,t]^2+ 2u(2,0)[x,t]}↵ {**4**, 8u[x,t]u(1,0)[x,t]-2u(3,0)[x,t]}↵ {**5**, 2(2u[x,t]^3-5u(1,0)[x,t]^2-6u[x,t]u(2,0)[x,t]+ u(4,0)[x,t])}↵ {**6**, -2(16u[x,t]^2*u(1,0)[x,t]-18u(1,0)[x,t]u(2,0)[x,t]- 8u[x,t]u(3,0)[x,t]+u(5,0)[x,t])}↵ <...cutting for convenience...>↵ {**10**, -2(256u[x,t]^4*u(1,0)[x,t]- 256u[x,t]^3*u(3,0)[x,t]+ 900u(1,0)[x,t]^2*u(3,0)[x,t]- 250u(3,0)[x,t]u(4,0)[x,t]-96u[x,t]^2*(18u(1,0)[x,t]u(2,0)[x,t]-u(5,0)[x,t])- 166u(2,0)[x,t]u(5,0)[x,t]+2u(1,0)[x,t]*(612u(2,0)[x,t]^2-35u(6,0)[x,t])- 16u[x,t]*(60u(1,0)[x,t]^3-68u(2,0)[x,t]u(3,0)[x,t]-40u(1,0)[x,t]u(4,0)[x,t]+ u(7,0)[x,t])+u(9,0)[x,t])}

Checkout on total derivatives shows that the effective are only odd ones because they are not total derivatives

{**0**, 0}↵ {1, 2}↵ {**2**, 0}↵ {**3**, -4u[x,t]}↵ {**4**, 0}↵ {**5**, 12u[x,t]^2-4u(2,0)[x,t]}↵ {**6**, 0}↵ {**7**, -4(10u[x,t]^3-5u(1,0)[x,t]^2-10u[x,t]u(2,0)[x,t]+u(4,0)[x,t])}↵ {**8**, 0}↵ {**9**, 4(35u[x,t]^4-70u[x,t]^2*u(2,0)[x,t]+ 21u(2,0)[x,t]^2+ 28u(1,0)[x,t]u(3,0)[x,t]+ 14u[x,t]*(-5u(1,0)[x,t]^2+u(4,0)[x,t])-u(6,0)[x,t])}↵ {**10**, 0}

The obtained result supposes that the initial system is integrable and it is reasonable to continue study to find the solutions. Also it makes possible to calculate Hamiltonian representations of the initial system and to get its representation in Lax pair as well, that is basis for the application of inverse scatering method.

This algorithm can be used also to classify the nonlinear dynamical systems on integrability. For example, studying the next equation (generalization of the fifth-order KdV system) [6]

$$u_t = u_{5x} + au^2u_x + bu_xu_{2x} + cuu_{3x} \tag{4}$$

One can conclude that only three types of equations pretend to be integrable (there is nondegenerated sequence of conservation laws):

$$u_t = u_{5x} + {}^{a^2}\!/_5\, u^2u_x + au_xu_{2x} + auu_{3x}$$
$$u_t = u_{5x} + {}^{a^2}\!/_5\, u^2u_x + {}^5\!/_2\, au_xu_{2x} + auu_{3x}$$
$$u_t = u_{5x} + {}^{3a^2}\!/_{10}\, u^2u_x + 2au_xu_{2x} + auu_{3x}$$

Decision to choose this method in preference to the others is justified by possibility to calculate hierarchy of conservation laws in algorithms not calculating the additional inessential conservation laws.

The result for system (4) was obtained without additional calculations and analysis by hand. Calculating conservation laws and their testing on adequacy to condition of conservation law (derivative on time from conservation law due to the initial system is total derivative with respect to x) it is easy to see that corresponding tests are:

{**0**, 0}↵ {**1**, 0}↵ {**2**, 0}↵ {**3**, -6/25(-2+α)(-1+5β)u(1,0)[x,t]u(2,0)[x,t]}↵ {**4**, 0}↵ {**5**, 1/25*(3-7α+2αβ^2+10β)(2u(2,0)[x,t]u(3,0)[x,t]+u(1,0)[x,t]u(4,0)[x,t])} etc.

There is substitution $\alpha = b/a,\ \beta = c/a^2$ in accordance to [6]. Since as conservation laws can serve only functions that are zero while checking, then, using the third and fifth pairs one can get three variants of combinations for the corresponding system to be integrable $\{\alpha = 2,\ \beta = 3/10\}$, $\{\alpha = 1,\ \beta = 1/5\}$, $\{\alpha = 5/2,\ \beta = 1/5\}$.
That coincides with the result [6], but it is obtained using the less quantity of operations and without testing all possible variants (as it is customary to do via methods using the examination of options).

Proposed algorithm can be applicated to classify the integrability of the nonlinear equations (1). There was studied integrability of KdV nonlinear equation's modification (4) via this method (f is arbitrary function): $u_t = -u_{xxx} + f(u)u_x$. After calculating densities of conservation laws and testing on accordance to the necessary condition (density is total derivative), the first-oder density can be represented as: $\frac{1}{3} f^{(IV)}(u)u_x^3 + f^{(III)}(u)u_x u_{2x}$. To obtain conservation law in this case the expression must be equal to zero at an abitrary function u. This can be valid if f is the next: $-c_1\left(u_x^6 \exp\left(-3uu_{2x}/u_x^2\right)\right)\Big/(27u_{2x}^3) + c_4u^2 + c_3u + c_2$, where c_i are arbitrary constants (method Mathematica was realized). Case with $c_2 = 1,\ c_1, c_3, c_4 = 0$ describes the linear equation, case $c_3 = 6,\ c_1, c_2, c_4 = 0$ describes standard KdV equation, case $c_4 = 6, c_1, c_2, c_3 = 0$ describes modified KdV equation. Nontrivial hierachy of conservation laws is calculated by the algorithm described in [10] in both first cases. Case with $c_1 = 1, c_2, c_3, c_4 = 0$ is so complicated in nonlinearity of equation that there is question about its physical intepretation.

Therefore, the next statement is true. Studied modification in the form

$$u_t = -u_{xxx} + f(u)u_x$$

has only well-known modifications integrable by ISM, in particular, modifications as: $u_t = -u_{xxx} + 6uu_x$ and $u_t = -u_{xxx} + 6u^2u_x$.

Also described algorithm can be applied with the corresponding modifications to the nonlinear evolutional dynamical systems.

Algorithm to calculate conservation laws of the nonlinear dynamical system is realized via SCA methods. Successful calculation of nondegenerated sequence of conservation laws is the first step to solving the nonlinear dynamical system by ISM.

Conclusions. The obtained results yield the application of the computer analytical methods to study the nonlinear dynamical systems in the mathematical physics. Hence scientific investigations in physics, chemistry, biology, economics need to use the nonlinear models, there is of current interest the conviction in application of solving methods. The developed procedure enables to study automatically the integrability of nonlinear dynamical systems for the unprofessional experts in this mathematical field and thus to obtain their results without exerting an efforts.

Reference

[1] *Olver P.J.* Applications of Lie Groups to Differential Equations - New York : Springer-Verlag, 1986.
[2] *Mitripolsky Ju.A., Bogolubov N.N., Prikarpatsky A.K., Samojlenko W.G.* Integrable dynamical systems: spectral and differential-geometric aspects - K.: Naukowa dumka, 1987. [in Russian]
[3] *Sanders J.A., Roelofs M.* An algorithmic approach to conservation laws using the 3-dimensional Heisenberg algebra. Tech. Rep. 2 RIACA. Amsterdam, 1994.
[4] *Sanders J.A., Wang J.P.* On the (non)existence of conservation laws for evolution equations. Tech. Rep. WS-445. Amsterdam : Vrije Universiteit, 1995.
[5] *Wolf T., Brand A., Mohammadzadeh M.* Computer algebra algorithms and routines for the computations of conservation laws and fixing of gauge in differential expressions. J.Symb.Comp., Vol. 27. - pp. 221-238, 1999.
[6] *Göktas Ü., Hereman W.* Symbolic computation of conserved densities for systems of nonlinear evolution equations. Technical Report MCS-96-06, Colorado School of Mines, 1996.
[7] *Ruo-Xia Yao, Zhi-Bin Li* CONSLAW: A Maple package to construct the conservation laws for nonlinear evolution equations. Applied Mathematics and Computation, Vol.173. - pp.616-635. 2006.
[8] *Prykarpats'kyy, A.K., Fil', B.M.* Category of topological jet manifolds and certain applications in the theory of nonlinear infinite-dimensional dynamical systems. Ukrainian Mathematical Journal, Vol.44(9), pp.1136-1148, 1992.
[9] *Mitropol'skii, Yu.O., Prikarpats'kii, A.K., Fil', B.M.* Some aspects of a gradient holonomic algorithm in the theory of integrability of nonlinear dynamic systems and computer algebra problems. Ukrainian Mathematical Journal, Vol.43(1), pp.63-74, 1991.
[10] *Fil', B.N., Prikarpatskii, A.K., Pritula, N.N.* Quantum lie algebra of currents - The universal algebraic structure of symmetries of completely integrable dynamical systems. Ukrainian Mathematical Journal, Vol.40(6), pp.645-649, 1988.
[11] *Samoilenko, V.G., Fil', B.N.* Algebras of symmetries of completely integrable dynamical systems. Ukrainian Mathematical Journal, Vol.40(2), pp.164-168, 1988.
[12] *P. Y. Pukach*, "On the unboundedness of a solution of the mixed problem for a nonlinear evolution equation at a finite time", Nonlinear Oscillations, Vol. 14, Is. 3, pp. 369-378, 2012.
[13] *Fil B., Prytula M.* Application of computer algebra methods to finding invariants of nonlinear dynamical systems. - Abstracts of the scientific and technical conference "Application of computational technology and mathematical methods in scientific and economic research", Kyiv, p. 136, 1991. [in Russian]

Алгоритм класифікації інтегрованості нелінійних динамічних систем методами комп'ютерної алгебри

Богдан Філь, Ярослав Пелех, Мирослава Вовк, Галина Берегова, Тетяна Магеровська, Павло Пукач

З використанням Wolfram Mathematica розроблений алгоритм класифікації інтегрованості нелінійних динамічних систем. З використанням розробленого алгоритму продемонстровано обчислення ієрархії законів збереження для нелінійних динамічних систем. Обчислені модифікації нелінійного рівняння Кортевега-де Фріза, інтегровані методом оберненої задачі розсіювання.

Received 28.02.2021